Vocabulary

$A \Rightarrow B$:

"A implies B" or "if A then B"

$A \Leftrightarrow B$:

"A is equivalent to B" or "A if and only if B"

"tautology" / "contradiction" :

always true / always false

"Which of the following..."

"Prove that..."

"Simplify..."

Propositional Logic is not enough

Consider:

"All students love discrete mathematics."

In propositional logic, expressed as:

"Kitchapun loves discrete maths" ∧ "Natthawut loves discrete maths"
∧ "Kridsana loves discrete maths" ∧ "Jittiya loves discrete maths"
∧ "Pannipa loves discrete maths" ∧ "Pimsiri loves discrete maths"
∧ ... (+ 70 more students!)

Difficult to prove anything!

So we will look at a more powerful type of logic...

Predicates

A *predicate* is a statement involving a variable.

Examples:

"$x > 3$"

"$x + y = z$"

"x loves discrete maths"

If P(x) represents the predicate "x loves discrete maths" then we can write the statement:

P(Ajahn Jaratsri)

And then we can determine its truth value:

P(Ajahn Jaratsri) → T

Predicates

A predicate with no variables is just a statement.

(Same as propositional logic)

Examples:

"$5 < 10$"

"Pimsiri loves discrete maths"

Predicates

Question 1:

If Q(x) is the statement "x < 5" then what are the truth values of Q(2) and Q(7)?

Question 2:

If R(x,y) is the statement "x * y > 20" then what are the truth values of R(4,6) and R(2,3)?

Quantification

Even with predicates, we still have long statements like:

P(Kitchapun) ∧ P(Natthawut) ∧ P(Kridsana) ∧ P(Jittiya) ∧ P(Pannipa) ∧ ...

The solution is *quantifiers*!

To express the statement "all students love discrete maths":

∀x P(x)

(meaning: for all x, the predicate P(x) is true)

Upside down A means "for all": ∀

Quantification

Example:

Let Q(x) be the statement "x > 10" then what is the truth value of ∀x Q(x)?

Q(3) is false, so ∀x Q(x) is false.

Same as saying "all numbers are larger than 10" -- which is obviously false.

Quantification

Question:

Let P(x) be the statement "x + 1 > x". What is the truth value of ∀x P(x)?

P(x) is true for all numbers, so ∀x P(x) is true.

Quantification

Another quantifier is for "there exists"...

$\exists x\ P(x)$

(meaning: there exists at least one x where P(x) is true)

If Q(x) is the statement "x cannot fly", then we can express the sentence "there exists a bird that cannot fly" as:

$\exists x\ Q(x)$

(where the domain of x is birds)

Quantification

Example:

Let Q(x) be the statement "x > 10" then what is the truth value of $\exists x\ Q(x)$?

Q(15) is true, so $\exists x\ Q(x)$ is true (at least one x is true).

Same as saying "there is at least one number that is greater than 10".

What does "at least" mean?

Quantification

Question:

Let P(x) be the statement "x - 1 > x". What is the truth value of $\exists x\ P(x)$?

P(x) is false for all numbers, so $\exists x\ P(x)$ is false.

Quantification

Usually, we define the *domain* that the quantifier applies to:

$\forall x \in \text{Students} : P(x)$

$\exists y \in \text{Animals} : Q(y)$

Quantification

Statement	When true?	When false?
$\forall x : P(x)$	P(x) is true for every x.	There is an x for which P(x) is false.
$\exists x : P(x)$	There is an x for which P(x) is true.	P(x) is false for every x.

Quantification

Quantifiers can be nested:

$$\forall x \in \mathbb{N} : \exists y \in \mathbb{N} : P(x,y)$$

(where P(x,y) is "$x < y$")

Meaning: "for all x in the natural numbers, there exists at least one y in the natural numbers where $x < y$ is true."

Quantification

Statement	When true?	When false?
$\forall x : \forall y : P(x,y)$	P(x,y) is true for every x,y.	There is an x,y for which P(x,y) is false.
$\forall x : \exists y : P(x,y)$	For every x there is a y for which P(x,y) is true.	There is an x such that P(x,y) is false for every y.
$\exists x : \forall y : P(x,y)$	There is an x for which P(x,y) is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x : \exists y : P(x,y)$	There is an x,y for which P(x,y) is true.	P(x,y) is false for every x,y.

Binding variables

In the statement $\exists x : (x + y = 1)$, we say:

"the variable x is *bound*"

but "the variable y is *free*"

All variables must either be bound or assigned a value for the statement to be valid.

More laws of logic

(∀x : T) → T
(∀x : F) → F
(∃x : T) → T
(∃x : F) → F

Assuming Q does not contain x:
∀x : (P(x) ∧ Q) → (∀x : P(x)) ∧ Q
∃x : (P(x) ∧ Q) → (∃x : P(x)) ∧ Q
∀x : (P(x) ∨ Q) → (∀x : P(x)) ∨ Q
∃x : (P(x) ∨ Q) → (∃x : P(x)) ∨ Q

De Morgan's laws for predicates:
¬∀x : P(x) → ∃x : ¬P(x)
¬∃x : P(x) → ∀x : ¬P(x)

More laws of logic

Quantifier laws — ระวัง !

∀x : (P(x) ∧ Q(x)) → (∀x : P(x)) ∧ (∀x : Q(x))
∃x : (P(x) ∨ Q(x)) → (∃x : P(x)) ∨ (∃x : Q(x))

(∀x : P(x)) ∨ (∀x : Q(x)) can be replaced by ∀x : (P(x) ∨ Q(x))
∃x : (P(x) ∧ Q(x)) can be replaced by (∃x : P(x)) ∧ (∃x : Q(x))

But not vice-versa!

Logical equivalence

Example:

Show that ¬ ∀x : (P(x) ⇒ Q(x)) is equivalent to ∃x : (P(x) ∧ ¬Q(x)) .

Proof:

¬ ∀x : (P(x) ⇒ Q(x)) → ∃x : ¬(P(x) ⇒ Q(x)) (De Morgan's law)
→ ∃x : ¬(¬(P(x) ∧ ¬Q(x))) (implication law)
→ ∃x : (P(x) ∧ ¬Q(x)) (double negation)

The End

That is *Logic* complete!