2. Sets

Discrete Mathematics

Set theory

The basic concept of a set:

> "By a set we shall understand any collection into a whole M of definite, distinct objects of our intuition or of our thought. These objects are called the elements of M." - Georg Cantor

Georg Cantor (1845-1918), a German mathematician, is the creator of modern set theory.

Set theory

Definition:

A *set* is an unordered collection of objects.

The objects in a set are called the *elements* or *members* of the set.

So, a set *contains* elements.

Set theory

Examples:

The set of teachers in CSIT contains Ajahn Jaratsri, Ajahn Antony, Ajahn Linda...

TeachersInCSIT = { Ajahn Jaratsri, Ajahn Antony, Ajahn Linda, ... }

The set of natural numbers...

N = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ... }

The set departments in the Faculty of Science:

ScienceDepartments = { CSIT, Mathematics, Physics, Biology, Chemistry }

Set theory

Anything can be a set...

FacultiesAtNU = { ScienceDepartments, HumantiesDepartments, MedicalSchool, ... }

= { { CSIT, Mathematics, Physics, Biology, Chemistry }, HumanitiesDepartments, MedicalSchool, ... }

An element in a set might also be a set. (Sets inside sets!)

Set theory

The empty set is a set that contains no elements.

Example:

The set of students who do not love discrete maths.

StudentsWhoDoNotLoveDiscreteMaths = { } = ∅

Special character for the empty set: ∅

Set theory

The set of odd natural numbers less than 10:

O = { 1, 3, 5, 7, 9 }

Sets are often described using the set builder notation.

For example, to describe the set of odd natural numbers less than 10:

O = { $x \in \mathbf{N}$ | x is odd and $x < 10$ }

Set theory

Common sets:

$\mathbf{N}$ = { 0, 1, 2, 3, ... }, the set of natural numbers

$\mathbf{Z}$ = { ..., -2, -1, 0, 1, 2, ... }, the set of integers

$\mathbf{Z}^+$ = { 1, 2, 3, ... }, the set of positive integers

$\mathbf{Q}$ = { p/q | $p \in Z$, $q \in Z$, and $q \neq 0$ }, the set of rational numbers

$\mathbf{R}$, the set of real numbers

Set theory

Definition:

Two sets are *equal* if and only if they have the same elements.

Let A and B be sets.

A and B are equal if and only if $\forall x : (x \in A \Leftrightarrow x \in B)$.

Written as: $A = B$

Set theory

Definition:

The set A is a *subset* of set B if and only if every element of A is also an element of B.

Written as: $A \subseteq B$

$A \subseteq B \rightarrow \forall x : (x \in A \Rightarrow x \in B)$

Set theory

For every set S,

$\varnothing \subseteq S$

and

$S \subseteq S$

are always true.

Set theory

Definition:

The set A is a *proper subset* of B if and only if A is a subset of B and A is not equal to B.

Written as: $A \subset B$

$A \subset B \equiv A \subseteq B \wedge A \neq B$

Set theory

Let X be all the students at NU and let Y be all the students studying computer science in CSIT.

Question: Which of the following are correct?

$Y \subseteq X$

$Y = X$

$Y \subset X$

Finite vs infinite sets

Examples of finite sets:

- Set of teachers in CSIT
- Set of boolean values { T, F }
- Set of natural numbers less than 100
- The empty set

Examples of infinite sets:

- Set of real numbers
- Set of odd natural numbers { 1, 3, 5, 7, ... }

Finite vs infinite sets

Definition:

Let S be a set. If there are exactly n distinct elements in S (where n is a natural number) then S is a *finite set*.

We say that n is the *cardinality* of S.

(Cardinality means 'number of elements')

The cardinality of S is written as: $| S |$

Finite vs infinite sets

Definition:

A set is an *infinite set* if it is not finite.

Finite vs infinite sets

Question 1:

Let E be the set of all even positive integers less than 20. Is E finite or infinite?

Question 2:

What is $|E|$?

Question 3:

Let F be the set of all real numbers greater than 0 and less than 10. Is F finite or infinite?

Power set

Definition:

Let S be a set. The *power set* of S is the set of all subsets of the set S.

The power set is written as: P(S)

Example:

The power set of {0, 1, 2} is...

$$P(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

Russell's Paradox

> There is only one barber in the town. Some of the men in the town shave themselves and the rest of the men are shaved by the barber. The barber shaves only the men who do not shave themselves.

Who shaves the barber?

Bertrand Russell (1872-1970) was a British philosopher and mathematician

In Logic:

$\forall x$: (x is shaved by the barber $\Leftrightarrow$ x does not shave himself)

Contradiction!

Russell's Paradox

Similarly...

A set can contain anything, including other sets. So imagine there is a *set of all sets*.

A set could even contain itself, for example: S = { a, b, c, S }. Most sets do not contain themselves. So now imagine the *set of all sets which do not contain themselves*.

The question is: does the set of all sets which do not contain themselves actually contain itself, or not?

Contradiction!

Russell's Paradox

What does it mean?

The axioms (fundamental laws) of set theory are not consistent.

But we will continue to use them!

Vocabulary

Set	- a collection of unordered objects
Element	- an object or member of a set
Contains	- is inside a set
Empty	- contains nothing
Finite	- has exactly x elements
Infinite	- opposite of finite
Cardinality	- number of elements in a set
Subset	- $A \subseteq B$
Proper subset	- $A \subset B$
Power set	- set of all possible subsets

Vocabulary

A set that contains no elements is called the empty set. It is finite and it has a cardinality of zero. It is a subset of all other sets.

Set operations

Like integers, like boolean logic, we can perform operations on sets.

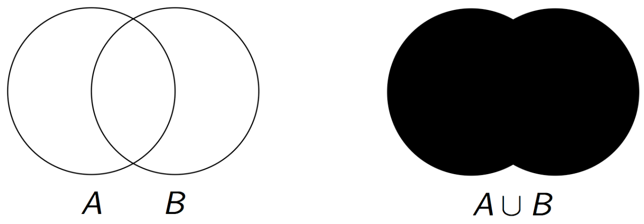
Four main operators we will use:

- Union
- Intersection
- Difference
- Complement

Set operations

Union

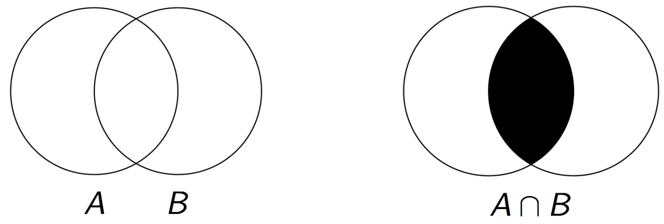
A ∪ B = { x | (x ∈ A) ∨ (x ∈ B) }



Set operations

Intersection

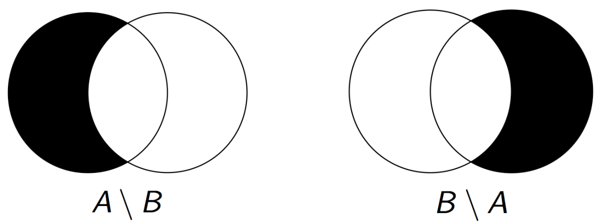
A ∩ B = { x | (x ∈ A) ∧ (x ∈ B) }



Set operations

Difference

A \ B = { x | (x ∈ A) ∧ (x ∉ B) }



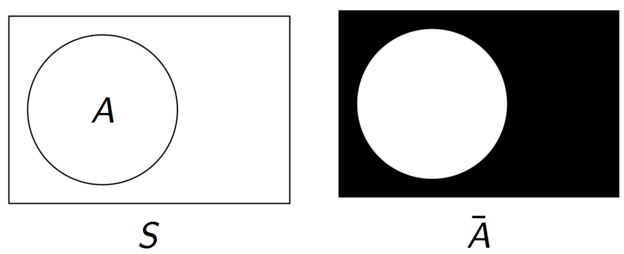
Set operations

Complement

Let S be the universal set, then

Ā = S \ A

or Ā = { x | x ∉ A }



Set operations

Examples:

A = {a, b, c} B = {c, a, f, g}

A ∪ B = {a, b, c, f, g}

A ∩ B = {a, c}

A \ B = {b}

B \ A = {f, g}

Laws of sets

$\overline{\overline{A}} = A$ (double complement law)

A ∩ A = A (idempotent law)
A ∪ A = A

A ∩ B = B ∩ A (commutative law)
A ∪ B = B ∪ A

(A ∩ B) ∩ C = A ∩ (B ∩ C) (associative law)
(A ∪ B) ∪ C = A ∪ (B ∪ C)

A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C) (distributive law)
A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Laws of sets

Prove A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C).

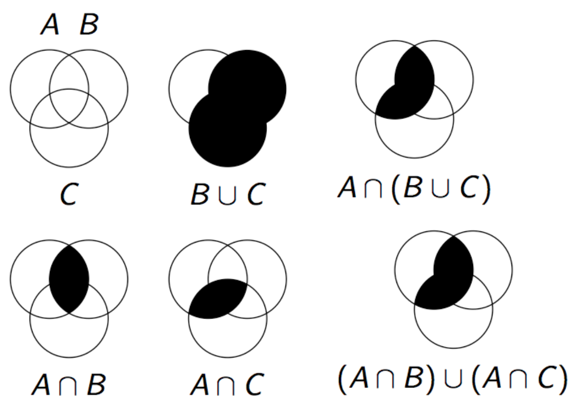
Consider any x.

x ∈ A ∩ (B ∪ C) → (x ∈ A) ∧ (x ∈ (B ∪ C))
→ (x ∈ A) ∧ (x ∈ B ∨ x ∈ C)
→ (x ∈ A ∧ x ∈ B) ∨ (x ∈ A ∧ x ∈ C)
→ (x ∈ A ∩ B) ∨ (x ∈ A ∩ C)
→ x ∈ (A ∩ B) ∪ (A ∩ C)

Hence A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

Laws of sets

Venn diagram of A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)



Laws of sets

De Morgan's laws:

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Also:

$A \cap B = \overline{\overline{A} \cup \overline{B}}$

$A \cup B = \overline{\overline{A} \cap \overline{B}}$

Laws of sets

Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Consider any $x \in S$.

$x \in \overline{A \cap B} \rightarrow x \notin A \cap B$

$\rightarrow \neg(x \in A \wedge x \in B)$

$\rightarrow (x \notin A) \vee (x \notin B)$

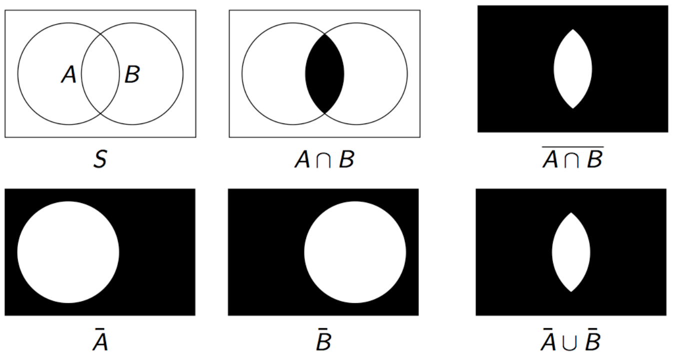
$\rightarrow (x \in \overline{A}) \vee (x \in \overline{B})$

$\rightarrow x \in (\overline{A} \cup \overline{B})$

Hence $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Laws of sets

Venn diagram of $\overline{A \cap B} = \overline{A} \cup \overline{B}$



Laws of sets

More laws:

Let $A \subseteq S$ for all A (S is the universe)

$A \cap S = A$	$A \cup \varnothing = A$	(identity law)
$A \cap \varnothing = \varnothing$	$A \cup S = S$	(annihilation law)
$A \cap \overline{A} = \varnothing$	$A \cup \overline{A} = S$	(law of excluded middle)
$A \cap (A \cup B) = A = A \cup (A \cap B)$		(absorption law)

Properties of powersets

Recall that P(S) is the powerset of S, meaning the set of all subsets of S.

E.g. $P(\{0,1\}) = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$

Some properties of P(S):

1. If S is finite, then P(S) is finite. If S has n elements, then P(S) has 2^n elements.
2. If S is infinite, then P(S) is infinite.
3. $P(A \cap B) = P(A) \cap P(B)$
4. In general, $P(A \cup B) \neq P(A) \cup P(B)$

Properties of powersets

Prove or disprove: For all A, B, $P(A) \cup P(B) \subseteq P(A \cup B)$?

TRUE

Proof. Consider any X.

$X \in P(A) \cup P(B) \rightarrow (X \in P(A)) \vee (X \in P(B))$
$\rightarrow (X \subseteq A) \vee (X \subseteq B)$
$\rightarrow (\forall x \in X : x \in A) \vee (\forall x \in X : x \in B)$
$\rightarrow \forall x \in X : (x \in A) \vee (x \in B)$
$\rightarrow \forall x \in X : (x \in A \cup B)$
$\rightarrow X \subseteq A \cup B$
$\rightarrow X \in P(A \cup B)$

Properties of powersets

Prove or disprove: For all A, B, $P(A \cup B) \subseteq P(A) \cup P(B)$?

FALSE

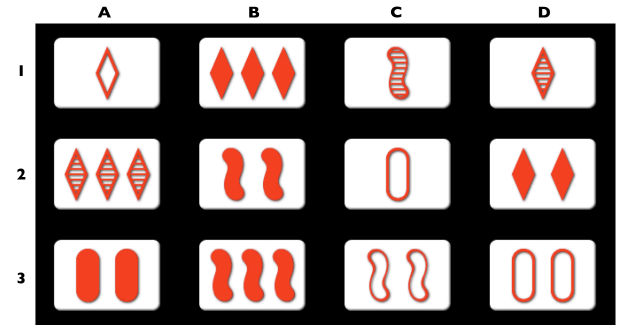
Proof.

Need to find A, B, X such that $X \in P(A \cup B)$ and $X \notin P(A) \cup P(B)$.

Let $A = \{0\}$, $B = \{1\}$. Let $X = \{0,1\}$.

$X \subseteq A \cup B = \{0,1\} \rightarrow X \in P(A \cup B)$
$X \notin P(A) = \{\varnothing,\{0\}\}$ and $X \notin P(B) = \{\varnothing,\{1\}\} \rightarrow X \notin P(A) \cup P(B)$

Set puzzle!



How to play: Three cards are a set if, for all features (shape, fill, count), the feature is the same on every card or the feature is distinct on every card. Can you find the 6 sets in the puzzle above?

http://thebreretons.com/setgame/?easy=color

Example: 

Sequences

Definition:

A *sequence* is an ordered collection of elements: $(x_1, x_2, \ldots, x_n)$

Example:

MySequence = (8,1,4)
AnotherSequence = (1,4,8,1)
MySequence ≠ AnotherSequence

A sequence is different from a set!

Order and repetitions are important: (x,y) ≠ (y,x) ≠ (y,x,x)

A sequence of length 2 is called an ordered pair: (x,y)

Cartesian product

(Another set operation...)

The cartesian product of sets A and B is the set of all ordered pairs (a, b), where $a \in A$, $b \in B$

$A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\}$

Example:

$\{1,2\} \times \{3,4\} = \{(1,3), (1,4), (2,3), (2,4)\}$

Rene Descartes (1596–1650) was a French philosopher and mathematician.

Cartesian product

More examples:

$\varnothing \times A = A \times \varnothing = \varnothing$ for any set A

{PapayaSalad} × {StickyRice} = {(PapayaSalad, StickyRice)}

{StickyRice} × {PapayaSalad} = {(StickyRice, PapayaSalad)}

$\{a, b, c\} \times \{d, e\} = \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$

$\mathbf{N}$ × Planets = { (n, x) | ($n \in \mathbf{N}$) ∧ (x ∈ Planets)} = {(5, Saturn), (23, Earth), . . .}

$\mathbf{N}^2 = \mathbf{N} \times \mathbf{N} = \{ (m,n) \mid m,n \in \mathbf{N} \}$

Cartesian product

Question 1: What is the cartesian product of {NexusOne, iPhone4, HTC} and {iOS, WindowsMobile, Android}?

Question 2: For any sets A,B, what is the cardinality of A × B?

Cartesian product

Distributive laws for cartesian product:

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

$(A \cap B) \times C = (A \times C) \cap (B \times C)$

$A \times (B \cup C) = (A \times B) \cup (A \times C)$

$(A \cup B) \times C = (A \times C) \cup (B \times C)$

$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

$(A \setminus B) \times C = (A \times C) \setminus (B \times C)$

Cartesian product

Example:

Prove $A \times (B \cup C)$ is equivalent to $(A \times B) \cup (A \times C)$.

Consider any (x,y).

$(x,y) \in A \times (B \cup C) \rightarrow (x \in A) \wedge (y \in (B \cup C))$

$\rightarrow (x \in A) \wedge (y \in B \vee y \in C)$

$\rightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$

$\rightarrow ((x,y) \in A \times B) \vee ((x,y) \in A \times C)$

$\rightarrow (x,y) \in (A \times B) \cup (A \times C)$

Hence $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Cartesian product

The cartesian product of sets $A_1, A_2, \ldots, A_n$ is the set of all ordered sequences$(a_1,a_2,\ldots,a_n)$, where $a_i \in A_i$ for all $i \in \{1,\ldots,n\}$

$A_1 \times A_2 \times \cdots \times A_n = \{ (a_1,a_2,\ldots,a_n) \mid \forall i \in \{1,2,\ldots,n\} : a_i \in A_i\}$

(The *cartesian product* of n sets is a set of *sequences* of length n.)

Vocabulary

Union

Difference

Intersection

Cartesian product

Sequence

Ordered pair

Complement