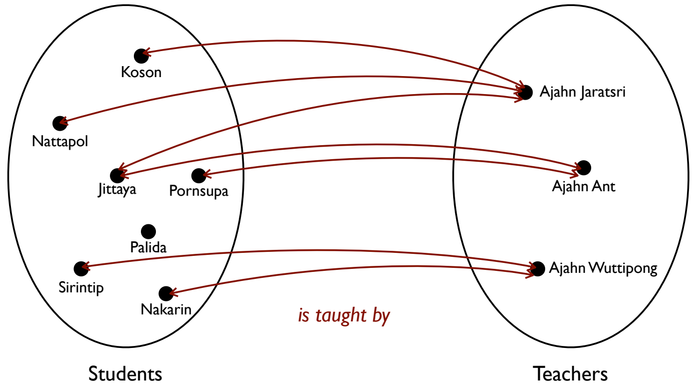
# 4. Functions

Discrete Mathematics



# Introduction

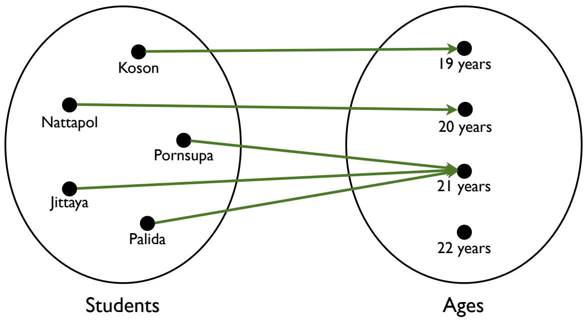
A *relation* can be any set of ordered pairs between two sets






# Introduction

A *function* is a special type of relation...




...where every member of A maps to exactly one member of B



# Introduction

Definition:

A *function* from set A to set B is a relation $R_f : A \leftrightarrow B$, where for every $a \in A$, there is a unique $b \in B$, such that a f b is true.

# (A bit more) Predicate logic

∃!x ∈ X : something

means

There exists a unique 'x' such that something.

∃ = there exists at least one
∃! = there exists only one



# Introduction

Definition:

A *function* from set A to set B is a relation $R_f$ : A ↔ B, where for every a ∈ A, there is a unique b ∈ B, such that a f b is true.

∀a ∈ A : ∃!b ∈ B : a f b



# Introduction

We write f : A → B for a function $R_f$ : A ↔ B, and say that:

f *maps* A onto B

A is the *domain*
B is the *range*

We write f(a) = b for a f b, and say that:

b is the *image* of a
(and a is the pre-image of b)



# Introduction

Examples:

Identity function $id_A$ : A → A $id_A$ = { (a,a) | a∈A } = $R_{=A}$

f : **N** → **N** f = { (m,n) ∈ $\mathbf{N}^2$ | $m^2$ = n } = {(0,0),(1,1),(2,4),(3,9),(4,16),...}

g : {0,1,2} → **N** $R_g$ = {(0,0),(1,1),(2,4)}

h : People → People y = h(x) ⇔ y is the mother of x

# Introduction

The following are NOT functions:

$r : \{0,1,2\} \to \mathbf{N}$ $r = \{(0,0),(1,1),(2,4),(2,8)\}$

(because the element 2 is mapped to 4 and 8)

$s : \mathbf{N} \to \mathbf{N}$ $s = \{(0,0),(1,1),(2,4)\}$

(because the domain is the natural numbers and we have only mapped 0,1,2)



# Composition

Let $R_p : A \leftrightarrow B$ and $R_q : B \leftrightarrow C$

If $R_p$ and $R_q$ are functions, then the composition $R_{p \circ q}$ is a function

Proof:

Consider any $x \in A$.
Since $R_p$ is a function, there is a unique $y \in B$, such that $x\ p\ y$.
Since $R_q$ is a function, there is a unique $z \in C$, such that $y\ q\ z$.
We have $(x\ p\ y) \wedge (y\ q\ z)$, so $x\ (p \circ q)\ z$.

Also, if $x\ (p \circ q)\ z'$, then $\exists y' : (x\ p\ y') \wedge (y'\ q\ z')$.
This can only happen if $y = y'$ and $z = z'$.

Therefore, for all $x \in A$, there is a unique $z \in C$, such that $x\ (p \circ q)\ z$, so $R_{p \circ q}$ is a function.



# Composition

$f : A \to B$ $g : B \to C$ $\forall a \in A : (f \circ g)(a) = g(f(a))$

The function $(f \circ g)$ *maps* A onto C.
The *domain* of $(f \circ g)$ is A.
The *range* of $(f \circ g)$ is C.

Examples:

Let $f, g : Z \to Z$ $f(n) = n+1$ $g(n) = n^2$

$(f \circ f)(n) = (n+1)+1 = n+2$
$(f \circ g)(n) = (n+1)^2 = n^2 + 2n + 1$
$(g \circ f)(n) = n^2 + 1$
$(g \circ g)(n) = (n^2)^2 = n^4$

Note: $f \circ g \neq g \circ f$



# Inversion

Let $R_p : A \leftrightarrow B$

Even if $R_p$ is a function, its inverse $R_{p-1}$ may not be a function

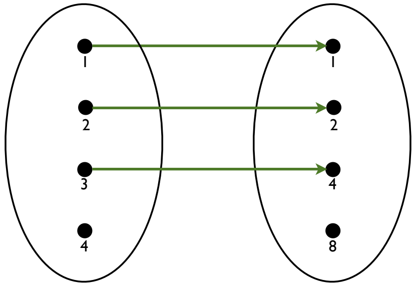
Examples:

$f : \mathbf{Z} \to \mathbf{Z}$ $f(n) = n+1$
$f_{-1}$ is a function: $f_{-1} : \mathbf{Z} \to \mathbf{Z}$ $f_{-1}(n) = n-1$

$g : \mathbf{N} \to \mathbf{N}$ $g(n) = n+1$
$R_{g-1}$ is not a function

$h : \mathbf{Z} \to \mathbf{Z}$ $h(n) = n^2$
$R_{h-1}$ is not a function

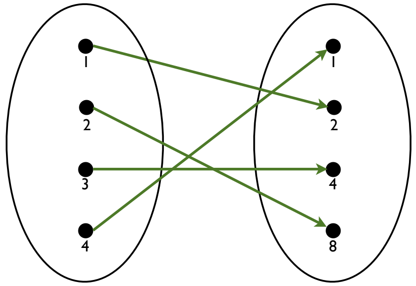
# Functions?




Question: Is this a function?



# Functions?

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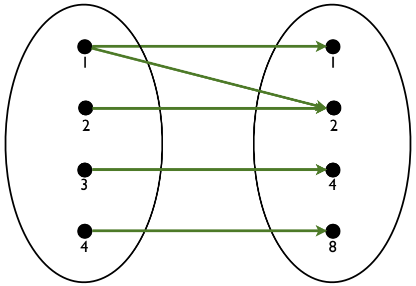
# Functions?

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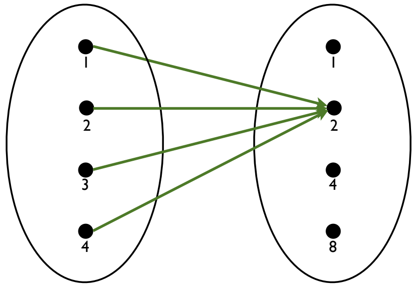
# Functions?

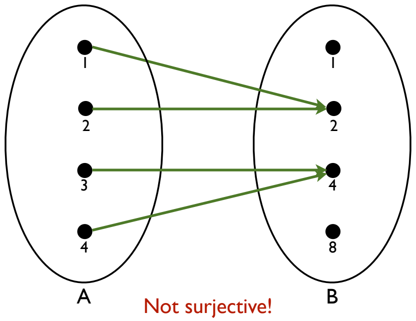



Question: Is this a function?

# Properties of functions

Function f : A → B is called *surjective* if its range f(A) is the whole of B

$$\forall b \in B : \exists a \in A : f(a) = b$$



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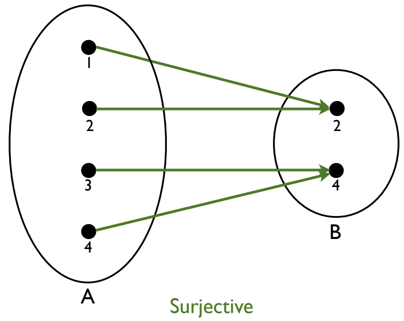
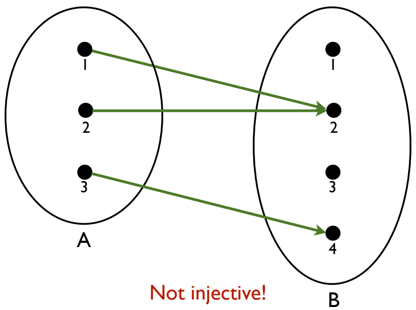





# Properties of functions

Function f : A → B is called *injective* if it maps different elements of A to different elements of B

$$\forall x,y : (f(x) = f(y)) \Rightarrow (x = y)$$



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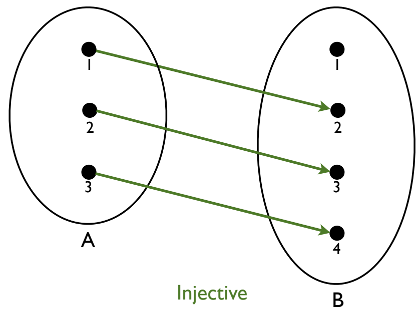


# Properties of functions

Examples:

f : Cards → {♠,♥,♣,♦}      f(x) = suit of x
Function f is surjective, but not injective

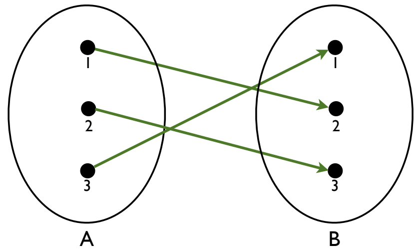
g : $\mathbf{N} \to \mathbf{N}$      $g(m) = m^2$
Function g is injective, but not surjective



# Properties of functions

Function f : A → B is called *bijective* if it is both surjective and injective.

We also say f is a one-to-one correspondence between A and B




Bijective



# Quiz

Which of the following are surjective or injective?

| Function | Surjective? | Injective? |
|---|---|---|
| f : $\mathbf{N} \to \mathbf{N}$<br>f(x) = 2x | | |
| g : $\mathbf{N} \to \mathbf{N}$<br>g(x) = 2 | | |
| h : $\mathbf{N} \to$ {2}<br>h(x) = 2 | | |
| i : {2} → {2}<br>i(x) = 2 | | |



# Vocabulary

A *function* is a special type of *relation*, written as

A *maps* onto B, where A is the *domain* and B is the *range*.

A function can be *surjective* or *injective*.

If it is both then it is called *bijective*.