6. Counting

Discrete Mathematics

Possible ways

In propositional logic, if you have A ∧ B then how many "possible ways" of choosing A and B are there?

A	B	A ∧ B
F	F	F
F	T	F
T	F	F
T	T	T

4 ways

(2 ways for A, 2 ways for B... 2 x 2 = 4)

Possible ways

An IP address (v4) is made up of 4 integers from 0-255. E.g. 10.31.23.27

How many possible IP addresses are there?

$256 \times 256 \times 256 \times 256 = 256^4 = 4{,}294{,}967{,}296$

(~4 million)

Product Rule

$|A \times B \times C| = |A| \times |B| \times |C|$

If you have x ways of doing task 1, y ways of doing task 2, and z ways of doing task 3 then you have x*y*z ways of doing all three

A = {1,2,3} B = {a,b} C = {a,b,g,d}

Question: How many 3-character words where 1st character is from A, 2nd from B and 3rd from C are there?

Answer: $|A| \times |B| \times |C| = 3 \times 2 \times 4 = 24$

Product Rule

All the rooms at Naresuan University are named using two letters and four numbers (e.g. SC2-313). How many possible room names are there?

26 x 26 x 10 x 10 x 10 x 10 = 676,000

Sum Rule

Question: There are 5 girls and 3 boys in a group. How many possible ways are there of choosing one person?

Answer: 5 + 3 = 8

Question: A student can choose a project from one of three lists. List A contains 23 projects, list B 15 projects, and list C 19 projects. How many possible projects are there to choose from?

Answer: 23 + 15 + 19 = 57

Sum Rule

If you have x ways of doing task 1 and y ways of doing task 2 then you have x + y ways of doing either task 1 or task 2

The Inclusion-Exclusion Principle

$|A \cup B| = |A| + |B| - |A \cap B|$

Question: How many 3-letter words start with "a" or end with "b"?

Word start with "a" = 26x26 = 676

Word end with "b" = 26x26 = 676

Word start with "a" and end with "b" = 26

Answer: 676 + 676 - 26 = 1,326

Pigeonhole Principle

In the Discrete Maths exam there are 61 possible grades (0-60).

There are more than 61 students in the class.

Therefore, there must be at least one grade that will be shared by at least 2 students.

(If there are 62 students in the class then it is not possible for everyone to have a different grade. Some students must have the same grade.)

Generalised Pigeonhole Principle

If N objects are assigned to k places, then at least one place must be assigned at least $\lceil N/k \rceil$ objects

Example: There are 69 students in this class and there are 12 months in the year, so at least $\lceil 69/12 \rceil = \lceil 5.75 \rceil = 6$ students share the same birth month.

Permutations

A permutation of a set S of objects is a sequence containing each object once.

An ordered arrangement of r distinct elements of S is an r-permutation.

The number of r-permutations of a set with n = |S| elements is:

$$P(n, r) = n! / (n - r)!$$

Permutations

Question: A karaoke system has1000 songs. How many ways are there of playing 20 different songs only once?

First song = 1000 ways

Second song = 999 ways

Third song = 998 ways ...

By product rule, total number of ways = 1000 x 999 x 998 x .... x 981

$$\frac{1000!}{(1000-20)!} = \frac{1000!}{980!}$$

Permutations with repetition

If the karaoke system can play 20 songs and it does not matter if they are repeated more than once, then

First song = 1000 ways

Second song = 1000 ways

Third song = 1000 ways ...

By product rule, total number of ways = 1000 x 1000 x ... = 1000^{20}

Combinations

A combination is like a permutation where the order is not important. (A subset not a sequence)

The number of r-combinations of a set with n = |S| elements is:

$$C(n,r) = P(n,r) / P(r,r) = n! / (r! \times (n-r)!)$$

Combinations

Question: How many distinct 7 card hands can be drawn from a standard 52 card pack?

The order of the cards does not matter.

Answer: C(52,7) = P(52,7) / P(7,7) = 52x51x50x49x48x47x46 / 7x6x5x4x3x2x1

= 133,784,560

Combinations with repetition

There is a further formula for combinations with repetition:

$$C(n+r-1,r)$$

Question: How many binary strings of length 16 contain exactly 4 one's?

Answer: C(16+4-1, 4) = C(19, 4)

Summary

PERMUTATIONS

Order Matters Repetition Allowed	$Possibilities = n^r$
Order Matters Repetition Not Allowed	$Possibilities = \frac{n!}{(n-r)!}$

COMBINATIONS

Order Doesn't Matter Repetition Allowed	$Possibilities = \frac{n!}{r!\,(n-r)!}$
Order Doesn't Matter Repetition Not Allowed	$Possibilities = \frac{(n+r-1)!}{r!\,(n-1)!}$

Sample questions

Suppose that a "word" is any string of seven letters of the alphabet, with repeated letters allowed.

1. How many words are there?
2. How many words end with the letter T?
3. How many words begin with R and end with T?
4. How many words begin with A or B?
5. How many words begin with A or end with B?
6. How many words begin with A or B and end with A or B?

Sample questions

7. How many words begin with A or B or end with A or B?
8. How many words begin with a vowel and end with a vowel?
9. How many words begin with a vowel or end with a vowel?
10. How many words begin with ABC in any order?
11. How many words have no vowels?
12. How many words have exactly one vowel?