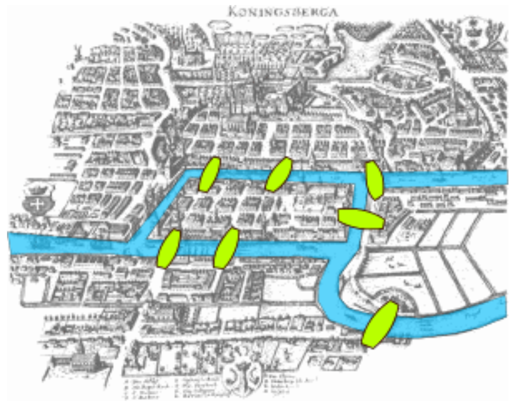
7. Graphs

Discrete Mathematics

Yipee!

Seven Bridges of Königsberg



Seven Bridges of Königsberg

During the 18th century (300 years ago) there was a city called Königsberg in Russia (now called Kaliningrad).

The city was 4 islands connected by 7 bridges.

The people liked to take long walks on Sundays.

They asked: “Is it possible to take a walk that crosses every bridge only once?”

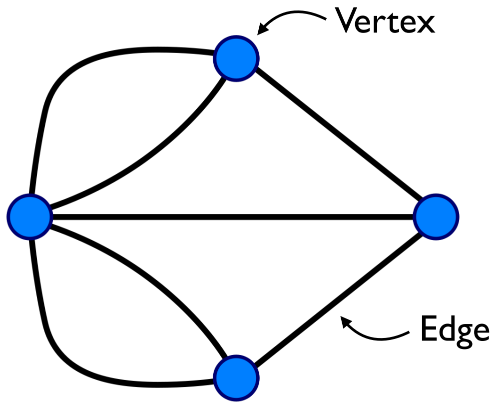
Seven Bridges of Königsberg

The Königsberg bridges problem was solved by Euler.

He represented the problem using vertices and edges.

His work became known as Graph Theory!

Seven Bridges of Königsberg



Set of vertices
+
Set of edges
=
Graph

Graphs

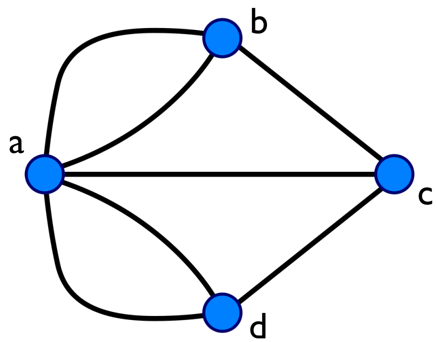
Definition:

A *graph* G = (V, E) consists of a set of *vertices* V and a set of *edges* E,

where each edge is a pair connecting two vertices

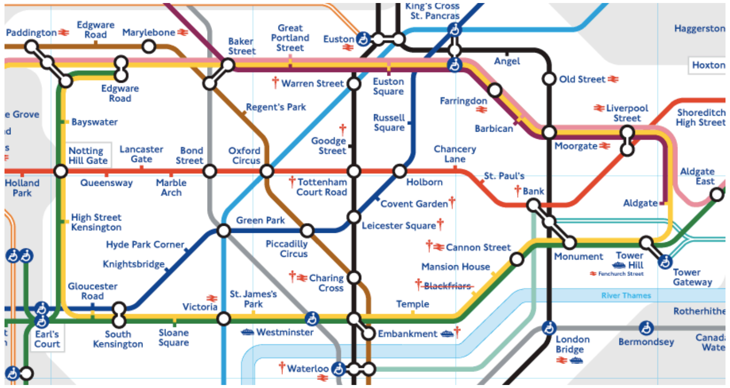
$E : V \Leftrightarrow V$ (E is a subset of $V \times V$).

Graphs



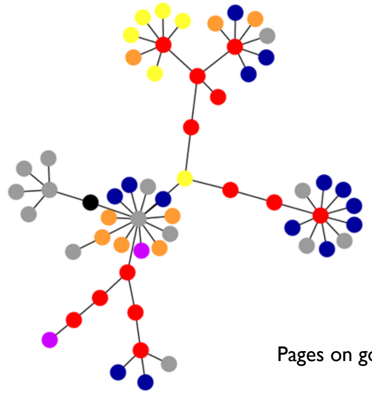
G = (V, E)
V = {a, b, c, d}
E = {(a,b), (b,a), (b,c), (a,c), (a,d), (d,a), (c,d)}

Examples of graphs



London Tube Map

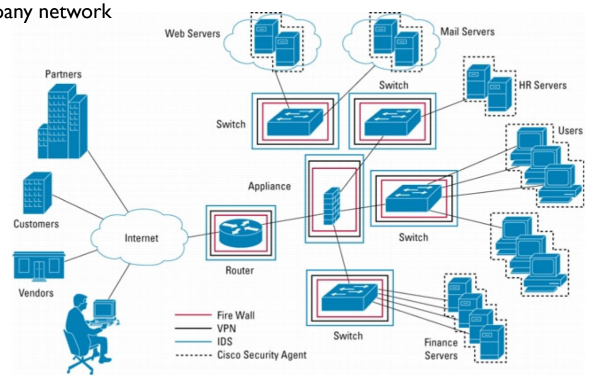
Examples of graphs



Pages on google.com

Examples of graphs

Company network

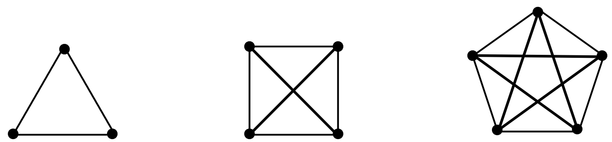


Properties of graphs

Definition:

A graph is *complete* if every vertex has an edge to every other vertex.

Examples:

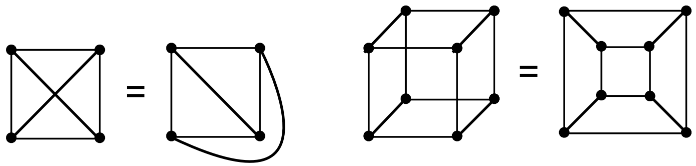


Properties of graphs

Definition:

A graph is called *planar* if it can be drawn without any lines crossing/intersecting.

Examples:

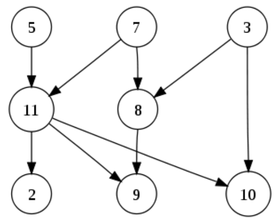


Directed graphs

So far, we have only seen *undirected graphs*.

A *directed graph* is a graph where the edges have a direction.

Example:



Directed graphs

Used to create *precedence graphs* of programs:

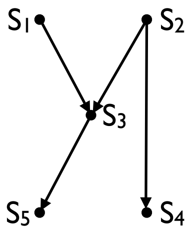
S_1 a = 0;

S_2 b = 1;

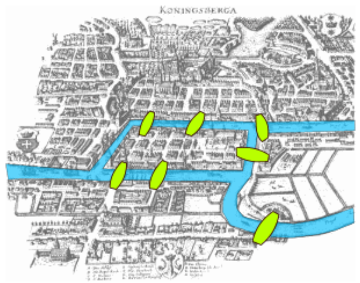
S_3 c = a + b;

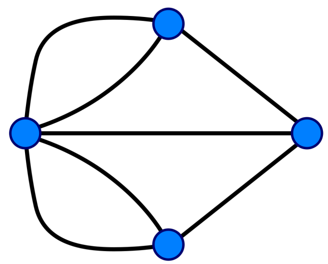
S_4 a = b;

S_5 b = c;



Back to bridges





Can you make a walk along every bridge/edge only once?

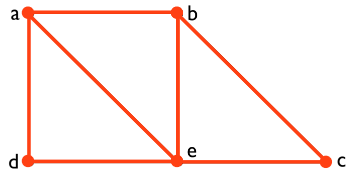
Back to bridges

Answer: No, there is no possible walk.

Why?

Euler paths and circuits

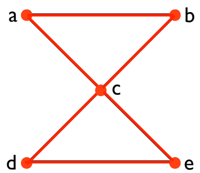
An *Euler path* is a walk in a graph that visits every **edge** exactly once.



Euler path: a, d, e, c, b, e, a, b

Euler paths and circuits

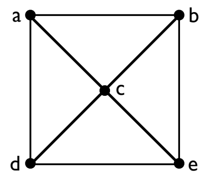
An *Euler circuit* is an Euler path that starts and ends at the same vertex.



Euler circuit: a, b, c, d, e, c, a

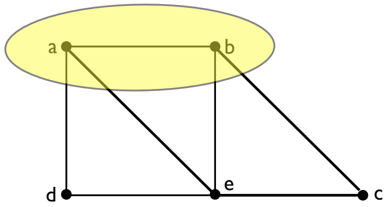
Euler paths and circuits

The following graph has no Euler path:



Euler paths and circuits

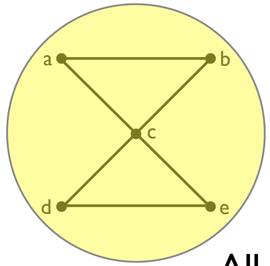
A graph is an Euler path if only two vertices have an odd number of edges.



Odd number of edges

Euler paths and circuits

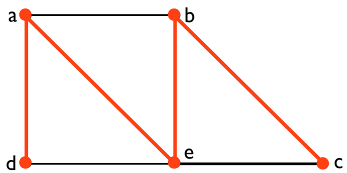
A graph is an Euler circuit if every vertex has an even number of edges.



All even edges

Hamilton paths and circuits

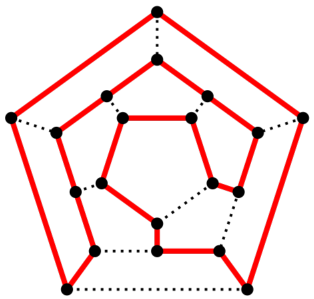
A *Hamilton path* is a walk in a graph that visits every **vertex** exactly once.



Hamilton path: d, a, e, b, c

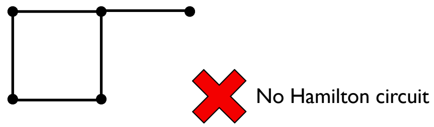
Hamilton paths and circuits

A *Hamilton circuit* is a Hamilton path that starts and ends at the same vertex.



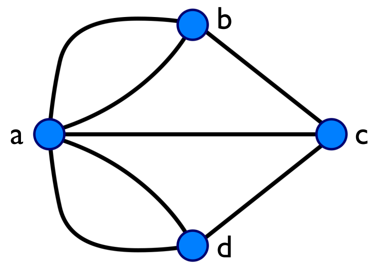
Hamilton paths and circuits

No simple method for determining whether a graph has a Hamilton path or circuit!



But if a vertex only has one edge then the graph cannot have a Hamilton circuit.

Back to bridges



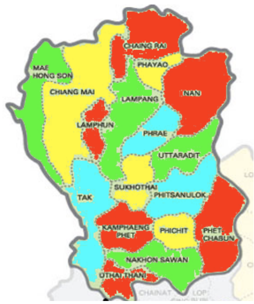
The Konigsberg bridges puzzle has no Euler path or circuit.

But, it has a Hamilton circuit: a, b, c, d

Graph colouring

Can you colour the provinces so that two neighbouring provinces do not have the same colour?

Graph colouring

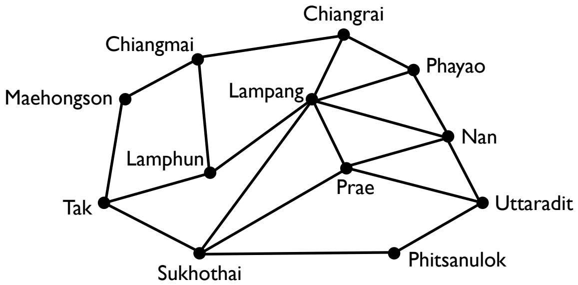


Yes

With 4 or more colours

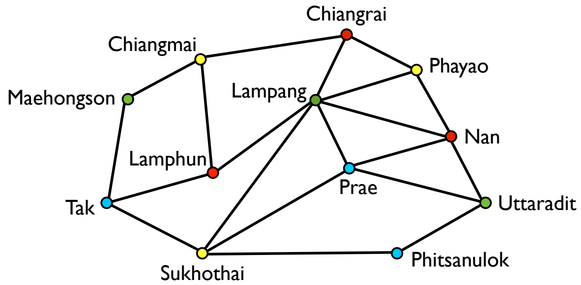
Graph colouring

Any map can be drawn as an planar graph:



Graph colouring

And then coloured...



The four colour theorem

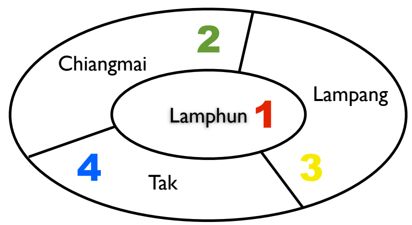
Theorem:

Any planar graph can be coloured using at most 4 colours.

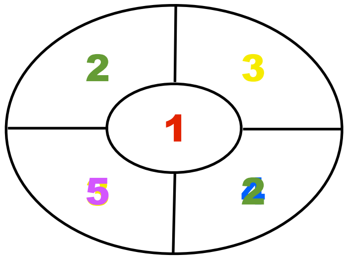
or

Any map can be coloured using at most 4 colours.

The four colour theorem



The four colour theorem



Only ever need 4 colours with planar graphs

(Sometimes less than 4!)

Vocabulary

Graph Vertex / vertices Edge / edges

Complete graph

Planar graph

Directed graph

Euler circuits & Euler paths

Hamilton circuits & Hamilton paths

Graph colouring

The End!